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ON THE NUMBER OF COMMUTATION CLASSES OF THE LONGEST ELEMENT IN THE SYMMETRIC GROUP

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ABSTRACT. Using the standard Coxeter presentation for the symmetric group S_n, two reduced expressions for the same group element are said to be commutation equivalent if we can obtain one expression from the other by applying a finite sequence of commutations. The resulting equivalence classes of reduced expressions are called commutation classes. How many commutation classes are there for the longest element in S_n?

Original proposer of the open problem: Donald E. Knuth
The year when the open problem was proposed: 1992 [11, §9]
Sponsor of the submission: Richard M. Green (University of Colorado Boulder)

A *Coxeter system* is a pair (W, S) consisting of a distinguished (finite) set S of generating involutions and a group

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

called a *Coxeter group*, where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st. Since the elements of S have order two, the relation $(st)^{m(s,t)} = e$ can be written to allow the replacement

$$\underbrace{sts\cdots}_{m(s,t)} \mapsto \underbrace{tst\cdots}_{m(s,t)}$$

which is called a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

Given a Coxeter system (W, S), a word $\mathsf{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$ in the free monoid S^* is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal among all expressions for w, the corresponding word is called a *reduced expression* for w. In this case, we define the *length* of w to be $\ell(w) = m$. According to [8], every finite Coxeter group contains a unique element of maximal length, which we refer to as the *longest element* and denote by w_0.

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FIGURE 1. Reduced expressions and the corresponding commutation classes for the longest element in S_4.

Let (W, S) be a Coxeter system and let $w \in W$. Then w may have several different reduced expressions that represent it. However, Matsumoto's Theorem [7, Theorem 1.2.2] states that every reduced expression for w can be obtained from any other by applying a finite sequence of commutations and braid moves.

Following [13], we define a relation $\sim$ on the set of reduced expressions for w. Let w and w' be two reduced expressions for w and define $\mathsf{w} \sim \mathsf{w}'$ if we can obtain w' from w by applying a single commutation. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a *commutation class*.

The Coxeter system of type A_{n-1} is generated by $S(A_{n-1}) = \{s_1, s_2, \ldots, s_{n-1}\}$ and has defining relations (i) $s_i s_i = e$ for all i; (ii) $s_i s_j = s_j s_i$ when $|i-j| > 1$; and (iii) $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$. The corresponding Coxeter group $W(A_{n-1})$ is isomorphic to the symmetric group S_n under the correspondence $s_i \mapsto (i,\ i+1)$. It is well known that the longest element in S_n is given in 1-line notation by

$$w_0 = [n, n-1, \ldots, 2, 1]$$

and that $\ell(w_0) = \binom{n}{2}$.

Let c_n denote the number of commutation classes of the longest element in S_n. The longest element w_0 in S_4 has length 6 and is given by the permutation $(1,4)(2,3)$. There are 16 distinct reduced expressions for w_0 while $c_4 = 8$. The 8 commutation classes for w_0 are given in Figure 1, where we have listed the reduced expressions that each class contains. Note that for brevity, we have written i in place of s_i.

In [12], Stanley provides a formula for the number of reduced expressions of the longest element w_0 in S_n. However, the following question is currently unanswered.

Open Problem. *What is the number of commutation classes of the longest element in S_n?*

To our knowledge, this problem was first introduced in 1992 by Knuth in Section 9 of [11], but not using our current terminology. A more general version of the problem appears in Section 5.2 of [9]. In the paragraph following the proof of Proposition 4.4 of [14], Tenner explicitly states the open problem in terms of commutation classes.

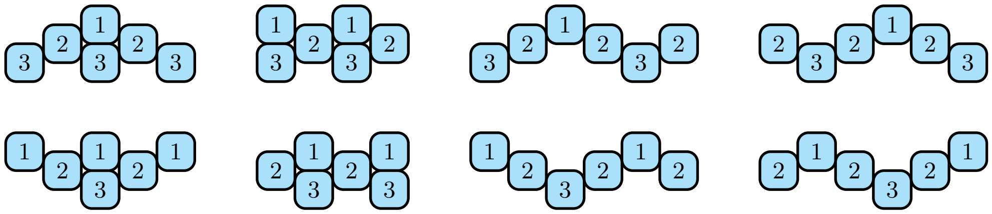

FIGURE 2. Heaps for the longest element in S_4.

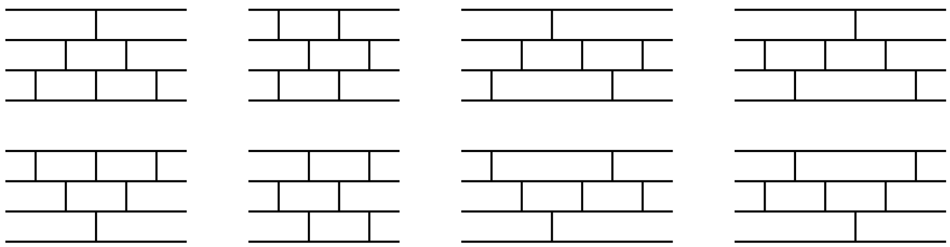

FIGURE 3. Minimal ladder lotteries corresponding to the primitive sorting networks on 4 elements.

According to sequence A006245 of The On-Line Encyclopedia of Integer Sequences [1], the first 10 values for c_n are $1, 1, 2, 8, 62, 908, 24698, 1232944, 112018190, 18410581880$. To date, only the first 15 terms are known. The current best upper-bound for c_n was obtained by Felsner and Valtr. They prove that for sufficiently large n, $c_n \leq 2^{0.6571n^2}$ [5, Theorem 2], although their result is stated in terms of arrangements of pseudolines.

The commutation classes of the longest element of the symmetric group are in bijection with a number of interesting objects. It turns out that c_n is equal to the number of

- heaps for the longest element in S_n [13, Proposition 2.2];
- primitive sorting networks on n elements [2, 10, 11, 15, 16];
- rhombic tilings of a regular $2n$-gon (where all side lengths of the rhombi and the $2n$-gon are the same) [3, 14];
- uniform oriented matroids of rank 3 on n elements [6, 9];
- arrangements of n pseudolines [4, 5, 11].

In Figure 2, we have drawn lattice point representations of the 8 heaps that correspond to the commutation classes for the longest element in S_4. Note that our heaps are sideways versions of the heaps that usually appear in the literature. The minimum ladder lotteries (or ghost legs) corresponding to the 8 primitive sorting networks on 4 elements are provided in Figure 3. The 8 distinct rhombic tilings of a regular octagon are depicted in Figure 4.

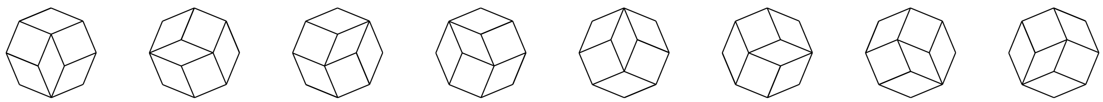

FIGURE 4. Rhombic tilings of a regular octagon.

Very little is known about the number of commutation classes of the longest element in other finite Coxeter groups.

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